Metamath and Metamath Zero

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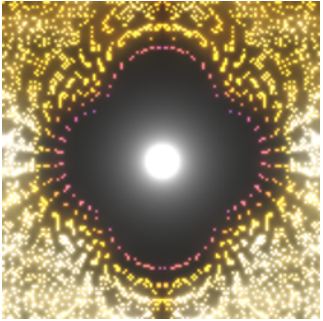
February 7, 2023

Who am I?



Github: `digama0`
Zulip: Mario Carneiro

- Postdoc in Logic at CMU
- Proof engineering since 2013
  - Metamath (maintainer)
  - Lean 3, Lean 4 mathlib (maintainer)
  - Dabbled in Isabelle, HOL Light, Coq, Mizar
  - Metamath Zero (author)
- Proved 37 of Freek's 100 theorems list in Metamath
- Lots of library code in `set.mm` and mathlib
- My PhD thesis was about Metamath Zero
- Say hi at `https://leanprover.zulipchat.com`

Part I: Metamath

Metamath is:

- A computer language for writing mathematical proofs
- A program `metamath.exe` to verify proofs in the Metamath language
- A library of completed proofs in a wide variety of axiomatic systems
  - `set.mm`: Over 40000 proofs deriving consequences of ZFC
    - Covers material in set theory, category theory, real analysis, calculus, number theory, algebra, topology, linear algebra, lattice theory, graph theory
    - 74 from Freek Wiedijk's 100 theorems list, which puts it 4th on the list behind HOL Light, Isabelle, and Coq
  - `iset.mm`: 10000 proofs in intuitionistic ZF
  - `nf.mm`: 5900 proofs in NF set theory
  - `ql.mm`: 1100 proofs in quantum logic
  - Other databases: `hol.mm`, `dtt.mm`, `peano.mm`, `miu.mm`

Metamath looks like: (set.mm)

```
$( Function with a domain of two different values.  (Contributed by FL,
   26-Jun-2011.)  (Revised by Mario Carneiro, 26-Apr-2015.) $)
fnprg $p ⊢ ( ( ( A e. V ∧ B e. W ) ∧ ( C e. X ∧ D e. Y ) ∧ A ≠ B )
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  SZQZTZJKUHZLSZTVRVSVMWAWEWCWFVMVNSZQZVOSZQZTZWERZWAWERZVMWKJSZKSZTZWEVMWHWN
  RZWJWORZUEZWKWPRVKVGWSVLVKWQWRVHVIWQVJJAEUFUGVIVHWRVJKBFUFUIUJUIWHWNWJWOUKU
  LJKUMUNWMWGWITZQZWERWLWAXAWEVTWTVNVOUMUOUPXAWKWEWGWIUQUPUTURVKVGWCWFRZVLVJV
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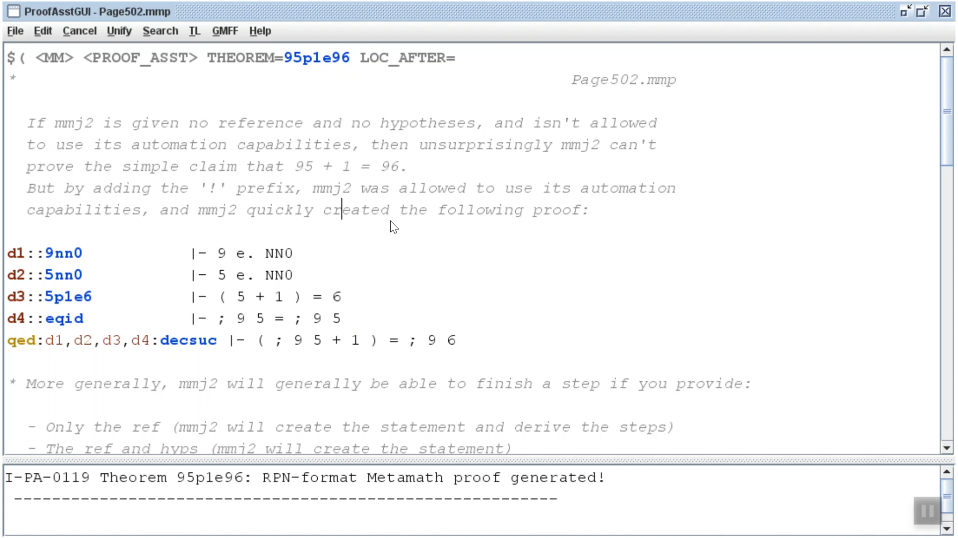
Metamath looks like: (mmj2)



Metamath looks like: (MPE)

Metamath Proof Explorer

< Previous Next >
Nearby theorems

Mirrors > Home > MPE Home > Th. List > ruc

Structured version Visualization version GIF version

Theorem ruc 14994

Description: The set of positive integers is strictly dominated by the set of real numbers, i.e. the real numbers are uncountable. The proof consists of lemmas ruclem1 14982 through ruclem13 14993 and this final piece. Our proof is based on the proof of Theorem 5.18 of [Truss] p. 114. See ruclem13 14993 for the function existence version of this theorem. For an informal discussion of this proof, see mmcomplex.html#uncountable. For an alternate proof see rucALT 14981. This is Metamath 100 proof #22. (Contributed by NM, 13-Oct-2004.)

Assertion

Ref	Expression
ruc	⊢ ℕ ≺ ℝ

Proof of Theorem ruc

Step	Hyp	Ref	Expression
1		reex 10049	..3 ⊢ ℝ ∈ V
2		nnssre 11046	..3 ⊢ ℕ ⊆ ℝ
3		ssdomg 8023	..3 ⊢ (ℝ ∈ V → (ℕ ⊆ ℝ → ℕ ≼ ℝ))
4	1, 2, 3	mp2 9	.2 ⊢ ℕ ≼ ℝ
5		ruclem13 14993	....5 ⊢ ¬ f:ℕ–onto→ℝ
6		f1ofo 6158	....5 ⊢ (f:ℕ–1-1-onto→ℝ → f:ℕ–onto→ℝ)
7	5, 6	mto 188	...4 ⊢ ¬ f:ℕ–1-1-onto→ℝ
8	7	nex 1731	..3 ⊢ ¬ ∃f f:ℕ–1-1-onto→ℝ
9		bren 7986	..3 ⊢ (ℕ ≈ ℝ ↔ ∃f f:ℕ–1-1-onto→ℝ)
10	8, 9	mtbir 313	.2 ⊢ ¬ ℕ ≈ ℝ
11		brsdom 8000	.2 ⊢ (ℕ ≺ ℝ ↔ (ℕ ≼ ℝ ∧ ¬ ℕ ≈ ℝ))
12	4, 10, 11	mpbir2an 955	1 ⊢ ℕ ≺ ℝ

Colors of variables: wff setvar class

Syntax hints: ¬ wn 3 ∃wex 1704 ∈ wcel 1990 Vcvv 3201 ⊆ wss 3576 *class class class* wbr 4657 –onto→wfo 5900 –1-1-onto→wf1o 5901 ≈ cen 7974 ≼ cdom 7975 ≺ csdm 7976 ℝcr 9957 ℕcn 11042

This theorem was proved from axioms: ax-mp 5 ax-1 6 ax-2 7 ax-3 8 ax-gen 1722 ax-4 1737 ax-5 1839 ax-6 1888 ax-7 1935 ax-8 1992 ax-9 1999 ax-10 2019 ax-11 2034 ax-12 2047 ax-13 2246 ax-ext 2603 ax-rep 4776 ax-sep 4786 ax-nul 4794 ax-pow 4848 ax-pr 4911 ax-un 6968 ax-cnex 10014 ax-resscn 10015 ax-1cn 10016 ax-icn 10017 ax-addcl 10018 ax-addrcl 10019 ax-mulcl 10020 ax-mulrcl 10021 ax-mulcom 10022 ax-addass 10023 ax-mulass 10024 ax-distr 10025 ax-i2m1 10026 ax-1ne0 10027 ax-1rid 10028 ax-rnegex 10029 ax-rrecex 10030 ax-cnre 10031 ax-pre-lttri 10032 ax-pre-lttrn 10033 ax-pre-ltadd 10034 ax-pre-mulgt0 10035 ax-pre-sup 10036

Metamath's good ideas

Metamath is not the most popular theorem prover,
but it has some good ideas that are not shared with its contemporaries.

What makes Metamath unique?

Metamath's good ideas

- Separate proof authoring from proof checking
- Have a simple spec for the logical core

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  - A good proof assistant is big and complex to give a good user experience
  - A good kernel is small and trustworthy
    (and ideally fast and not resource-intensive)

Separate proof authoring from proof checking

- Metamath stores *proofs*, not *proof scripts*

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  SZQZTZJKUHZLSZTVRVSVMWAWEWCWFVMVNSZQZVOSZQZTZWERZWAWERZVMWKJSZKSZTZWEVMWHWN
  RZWJWORZUEZWKWPRVKVGWSVLVKWQWRVHVIWQVJJAEUFUGVIVHWRVJKBFUFUIUJUIWHWNWJWOUKU
  LJKUMUNWMWGWITZQZWERWLWAXAWEVTWTVNVOUMUOUPXAWKWEWGWIUQUPUTURVKVGWCWFRZVLVJV
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- Checking metamath proofs is massively faster than checking Lean, Coq, Isabelle, HOL Light proofs
  - The classic verifier `metamath.exe` checks `set.mm`, a library on the same order of magnitude as Lean mathlib, in 8 seconds

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  SZQZTZJKUHZLSZTVRVSVMWAWEWCWFVMVNSZQZVOSZQZTZWERZWAWERZVMWKJSZKSZTZWEVMWHWN
  RZWJWORZUEZWKWPRVKVGWSVLVKWQWRVHVIWQVJJAEUFUGVIVHWRVJKBFUFUIUJUIWHWNWJWOUKU
  LJKUMUNWMWGWITZQZWERWLWAXAWEVTWTVNVOUMUOUPXAWKWEWGWIUQUPUTURVKVGWCWFRZVLVJV
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  - The classic verifier `metamath.exe` checks `set.mm`, a library on the same order of magnitude as Lean mathlib, in 8 seconds
  - An optimized metamath verifier has achieved the same feat in 0.9 seconds

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Metamath's good ideas

- Separate proof authoring from proof checking
- Have a simple spec for the logical core

Have a simple spec for the logical core

- Metamath has a prose specification in the Metamath book

The next section contains the complete specification of the Metamath language. It serves as an authoritative reference and presents the syntax in enough detail to write a parser and proof verifier. The specification is terse and it is probably hard to learn the language directly from it, but we include it here for those impatient people who prefer to see everything up front before looking at verbose expository material. Later sections explain this material and provide examples. We will repeat the definitions in those sections, and you may skip the next section at first reading and proceed to Section 4.2 (p. 116).

4.1 Specification of the Metamath Language

> *Sometimes one has to say difficult things, but one ought to say them as simply as one knows how.*
>
> G. H. Hardy[2]

4.1.1 Preliminaries

A Metamath **database** is built up from a top-level source file together with any source files that are brought in through file inclusion commands (see below). The only characters that are allowed to appear in a Metamath source file are the 94 non-whitespace printable ASCII characters, which are digits, upper and lower case letters, and the following 32 special characters:

```
! " # $ % & ' ( ) * + , - . / :
; < = > ? @ [ \ ] ^ _ ` { | } ~
```

plus the following characters which are the "white space" characters: space (a printable character), tab, carriage return, line feed, and form feed. We will use `typewriter` font to display the printable characters.

A Metamath database consists of a sequence of three kinds of **tokens** separated by **white space** (which is any sequence of one or more white space characters). The set of **keyword** tokens is `${`, `$}`, `$c`, `$v`, `$f`, `$e`, `$d`, `$a`, `$p`, `$.`, `$=`, `$(`, `$)`, `$[`, and `$]`. The last four are called **auxiliary** or preprocessing keywords. A **label** token consists of any combination of letters, digits, and the characters hyphen, underscore, and period. A **math symbol** token may consist of any combination of the 93 printable standard ASCII characters other than space or `$` . All tokens are case-sensitive.

[2] As quoted in [16], p. 273.

Have a simple spec for the logical core

- Metamath has a prose specification in the Metamath book
  - The full spec is 28 pages (with lots of explanation and examples)

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A Metamath database consists of a sequence of three kinds of **tokens** separated by **white space** (which is any sequence of one or more white space characters). The set of **keyword** tokens is `${`, `$}`, `$c`, `$v`, `$f`, `$e`, `$d`, `$a`, `$p`, `$.`, `$=`, `$(`, `$)`, `$[`, and `$]`. The last four are called **auxiliary** or preprocessing keywords. A **label** token consists of any combination of letters, digits, and the characters hyphen, underscore, and period. A **math symbol** token may consist of any combination of the 93 printable standard ASCII characters other than space or `$` . All tokens are case-sensitive.

[2] As quoted in [16], p. 273.

Have a simple spec for the logical core

- Metamath has a prose specification in the Metamath book
  - The full spec is 28 pages (with lots of explanation and examples)
- There is an emphasis on parsimony

The next section contains the complete specification of the Metamath language. It serves as an authoritative reference and presents the syntax in enough detail to write a parser and proof verifier. The specification is terse and it is probably hard to learn the language directly from it, but we include it here for those impatient people who prefer to see everything up front before looking at verbose expository material. Later sections explain this material and provide examples. We will repeat the definitions in those sections, and you may skip the next section at first reading and proceed to Section 4.2 (p. 116).

4.1 Specification of the Metamath Language

Sometimes one has to say difficult things, but one ought to say them as simply as one knows how.

G. H. Hardy[2]

4.1.1 Preliminaries

A Metamath **database** is built up from a top-level source file together with any source files that are brought in through file inclusion commands (see below). The only characters that are allowed to appear in a Metamath source file are the 94 non-whitespace printable ASCII characters, which are digits, upper and lower case letters, and the following 32 special characters:

```
! " # $ % & ' ( ) * + , - . / :
; < = > ? @ [ \ ] ^ _ ` { | } ~
```

plus the following characters which are the "white space" characters: space (a printable character), tab, carriage return, line feed, and form feed. We will use `typewriter` font to display the printable characters.

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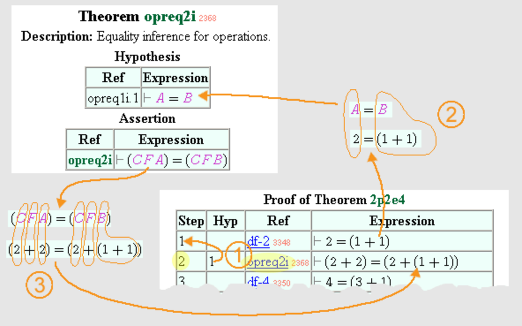
  - The full spec is 28 pages (with lots of explanation and examples)
- There is an emphasis on parsimony
- The HTML documentation is full of pages of introductory material which assumes no mathematical background



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    - A metamath verifier was adapted to prove the best known lower bound on the smallest unprovable-in-ZFC busy beaver number[1]

[1] `https://github.com/sorear/metamath-turing-machines`

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- Metamath has also been used for machine learning (Holophrasm, GPT-f)

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Metamath for AI/ML/ATP applications

- Metamath is a very friendly language for bulk processing, because it has such a simple grammar and few core concepts
  - In many cases you can get relevant and accurate information about theorem structure using regexes
  - There is only one kind of proof step (a theorem application), so proofs are just trees of applications and verification is uniform
- It also has a large body of human-curated mathematics, which is good for training and testing automated provers
- Verification and processing is quite fast, so the bottleneck is usually the external processing (the ATP, ML training etc)

Part II: Metamath Zero

Deficiencies of Metamath

- Metamath tries to simultaneously serve the human reader and the computer verifier, but they have divergent needs

```
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   26-Jun-2011.)  (Revised by Mario Carneiro, 26-Apr-2015.) $)
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  ( wcel wa wne w3a cop cpr wfun cdm wceq wfn funprg dmpropg 3ad2ant2 df-fn
  sylanbrc ) AEIBFIJZCGIDHIJZABKZLACMBDMNZOUGPABNZQZUGUHRABCDEFGHSUEUDUIUFACB
  DGHTUAUGUHUBUC $.

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  RZWJWORZUEZWKWPRVKVGWSVLVKWQWRVHVIWQVJJAEUFUGVIVHWRVJKBFUFUIUJUIWHWNWJWOUKU
  LJKUMUNWMWGWITZQZWERWLWAXAWEVTWTVNVOUMUOUPXAWKWEWGWIUQUPUTURVKVGWCWFRZVLVJV
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  - Metamath has a reputation for having no automation as a result
  - Existing MM proof assistants are certainly lacking in small scale automation compared to HOL light, Isabelle, Coq, Lean

```
$( Function with a domain of two different values.  (Contributed by FL,
   26-Jun-2011.)  (Revised by Mario Carneiro, 26-Apr-2015.) $)
fnprg $p ⊢ ( ( ( A e. V ∧ B e. W ) ∧ ( C e. X ∧ D e. Y ) ∧ A ≠ B )
      → { <. A , C >. , <. B , D >. } Fn { A , B } ) $=
  ( wcel wa wne w3a cop cpr wfun cdm wceq wfn funprg dmpropg 3ad2ant2 df-fn
  sylanbrc ) AEIBFIJZCGIDHIJZABKZLACMBDMNZOUGPABNZQZUGUHRABCDEFGHSUEUDUIUFACB
  DGHTUAUGUHUBUC $.

$( Function with a domain of three different values.  (Contributed by
   Alexander van der Vekens, 5-Dec-2017.) $)
fntpg $p ⊢ ( ( ( X e. U ∧ Y e. V ∧ Z e. W )
             ∧ ( A e. F ∧ B e. G ∧ C e. H )
             ∧ ( X ≠ Y ∧ X ≠ Z ∧ Y ≠ Z ) )
         → { <. X , A >. , <. Y , B >. , <. Z , C >. } Fn { X , Y , Z } ) $=
  ( wcel w3a wne cop cdm wceq csn cun ctp wfun wfn funtpg wa dmsnopg 3ad2ant1
  cpr 3ad2ant2 jca uneq12 syl df-pr syl6eqr dmeqi eqeq1i dmun sylibr 3ad2ant3
  bitri uneq12d df-tp eqtri 3eqtr4g df-fn sylanbrc ) JDMKHMLIMZNZAGOBFOCEOZNZ
  ZJKOLOZNZJKOLOZNZJKOLOSNVQNVPSNCDPJKPLVOPSNVLPQZKFPLLZKRLZKRJRPNP
  SZQZTZJKUHZLSZTVRVSVMWAWEWCWFVMVNSZQZVOSZQZTZWERZWAWERZVMWKJSZKSZTZWEVMWHWN
  RZWJWORZUEZWKWPRVKVGWSVLVKWQWRVHVIWQVJJAEUFUGVIVHWRVJKBFUFUIUJUIWHWNWJWOUKU
  LJKUMUNWMWGWITZQZWERWLWAXAWEVTWTVNVOUMUOUPXAWKWEWGWIUQUPUTURVKVGWCWFRZVLVJV
  HXBVILCGUFUSUIVAVRVTWBTZQWDVQXCVNVOVPVBUOVTWBUQVCJKLVBVDVQVSVEVF $.

${
  fntp.1 $e ⊢ A e. _V $.
  fntp.2 $e ⊢ B e. _V $.
  fntp.3 $e ⊢ C e. _V $.
  fntp.4 $e ⊢ D e. _V $.
  fntp.5 $e ⊢ E e. _V $.
  fntp.6 $e ⊢ F e. _V $.
  $( A function with a domain of three elements.  (Contributed by NM,
     14-Sep-2011.)  (Revised by Mario Carneiro, 26-Apr-2015.) $)
  fntp $p ⊢ ( ( A ≠ B ∧ A ≠ C ∧ B ≠ C )
        → { <. A , D >. , <. B , E >. , <. C , F >. } Fn { A , B , C } ) $=
    ( wne w3a cop ctp wfun cdm wceq wfn funtp dmtpop a1i df-fn sylanbrc ) ABM
    ACMBCMNZADOBEOCFOPZQUGRABCPZSUGUHTABCDEFGHIJKLUAUIUFADBECFJKLUBUCUGUHUDU
    E $.
$}
```

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  - Have a metaprogramming language for the front end (a tactic language)
- Give it a more modern-looking syntax
  - shamelessly borrowed from Lean

Introduction to Metamath C

Software without bugs is possible

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→ Software correctness can be proved by mathematical proof (“deductive verification”)

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Metamath C is a language for writing verified programs.

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- MMC: A proof-producing compiler
  - A programming language for producing (x86) programs with a proof of correctness

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- The MM0 verifier is written in MMC[2] – bootstrap!

[2]Currently the MMC verifier for MM0 is not finished, but there are multiple other MM0 verifiers so it can already be used without the bootstrap.

A simple MM0 file: propositional logic

```
delimiter $ ( ~ $  $ ) $;
strict provable sort wff;
term im (a b: wff): wff; infixr im: $->$ prec 25;
term not (a: wff): wff; prefix not: $~$ prec 40;

-- The Lukasiewicz axioms for propositional logic
axiom ax_1 (a b: wff): $ a -> b -> a $;
axiom ax_2 (a b c: wff):
  $ (a -> b -> c) -> (a -> b) -> a -> c $;
axiom ax_3 (a b: wff):
  $ (~a -> ~b) -> b -> a $;
axiom ax_mp (a b: wff):
  $ a -> b $ >
  $ a $ >
  $ b $;

-- Assert that `P -> P` is provable
theorem id (P: wff): $ P -> P $;
```

Peano arithmetic

```
... -- predicate logic

--| The sort of natural numbers, or nonnegative integers.
sort nat;

--| ‘0‘ is a natural number.
term d0: nat; prefix d0: $0$ prec max;
--| The successor operation: ‘suc n‘ is a natural number when ‘n‘ is.
term suc (n: nat): nat;

--| Zero is not a successor. Axiom 1 of Peano Arithmetic.
axiom sucne0 (a: nat): $ suc a != 0 $;
--| The successor function is injective. Axiom 2 of Peano Arithmetic.
axiom sucinj (a b: nat): $ suc a = suc b <-> a = b $;
--| The induction axiom of Peano Arithmetic. If ‘p(0)‘ is true,
--| and ‘p(x)‘ implies ‘p(suc x)‘ for all ‘x‘, then ‘p(x)‘ is true for all ‘x‘.
axiom induction {x: nat} (p: wff x):
  $ [ 0 / x ] p -> A. x (p -> [ suc x / x ] p) -> A. x p $;
```

Peano arithmetic

```
--| Addition of natural numbers, a primitive term constructor in PA.
term add (a b: nat): nat; infixl add: $+$ prec 64;
--| Multiplication of natural numbers, a primitive term constructor in PA.
term mul (a b: nat): nat; infixl mul: $*$ prec 70;

--| Addition respects equality.
axiom addeq (a b c d: nat): $ a = b -> c = d -> a + c = b + d $;
--| Multiplication respects equality.
axiom muleq (a b c d: nat): $ a = b -> c = d -> a * c = b * d $;
--| The base case in the definition of addition.
axiom add0 (a: nat): $ a + 0 = a $;
--| The successor case in the definition of addition.
axiom addS (a b: nat): $ a + suc b = suc (a + b) $;
--| The base case in the definition of multiplication.
axiom mul0 (a: nat): $ a * 0 = 0 $;
--| The successor case in the definition of multiplication.
axiom mulS (a b: nat): $ a * suc b = a * b + a $;
```

Peano arithmetic

- Peano arithmetic is a very simple axiomatic system, but also quite expressive

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We define:

- Propositional logic
- Predicate logic
- Class theory
- $+, -, *, /$, mod, gcd
- even, odd, disjoint sums
- ordered pairs, cartesian product
- finite functions, class functions
- Integers: $+, -, *, /$, mod
- Bitwise operators
- Recursion, exponentiation
- Lists
- Set operators
- finite sets, finite set theory
- cardinality
- List ops: length, append, repeat, reverse, map, join, filter, zip, …

Metaprogramming with MM1

- MM1 comes with a metaprogramming language based on Scheme

```
do {
  (display "hello world")      -- hello world
  {2 + 2}                      -- 4
  (def x 5)
  {x + x}                      -- 10
  (def (f y) {y + y})
  (f 3)                        -- 6
  (def (fact x)
    (if {x = 0}
      1
      {x * (fact {x - 1})}))
  (fact 5)                     -- 120
};
```

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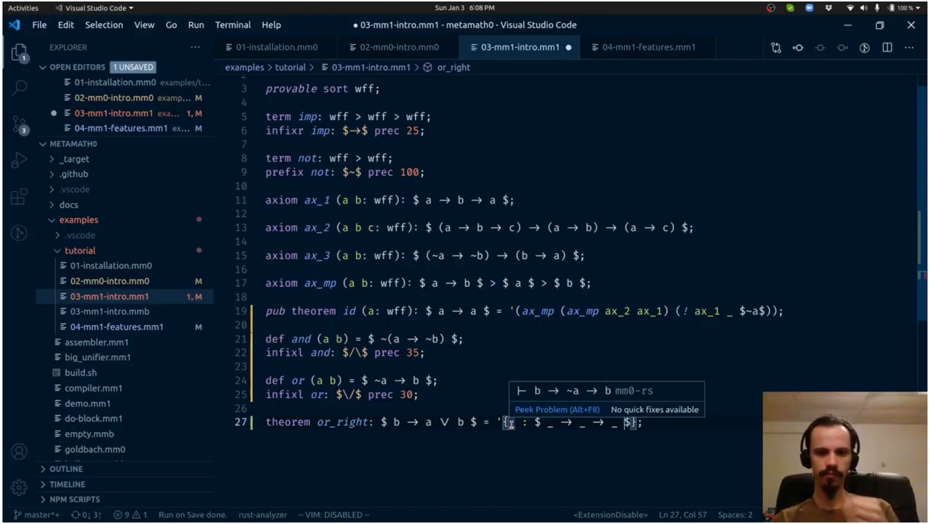
  - `theorem _` means it is an example
  - `norm_num` is the tactic which proves the theorem
  - `,19` calls a preprocessor to render 19 as a term. The actual theorem proved is:

```
theorem _: $ (x1 :x x3) * (x7 :x x8) + x2 = (x8 :x xe :x xa) $;
```

  that is, `0x13` · `0x78` + `0x2` = `0x8ea` which is the theorem written in hexadecimal

Working with MM1



From “Metamath Zero (MM0/MM1) tutorial”, `https://youtu.be/A7WfrW7-ifw`

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- To interpret IO, a model of the (Linux) operating system
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- The ELF file format (the linux equivalent of `.exe`)

Metamath C

- This is everything we need to state the correctness theorem for a compiled program:

Program correctness

Program P is correct to specification T if for every initial state $s \in \mathrm{init}(P)$, all nondeterministic evaluations do not cause undefined behavior, and after reaching a final state $s \rightsquigarrow^* s'$, if s' is a successful exit state and $\mathrm{input\_consumed}(s') = I$ and $\mathrm{output\_produced}(s') = O$, then $T(I, O)$ is true.

- Red: definitions from `x86.mm0`
- Blue: the user specification

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- The Metamath C compiler produces theorems of this form.

Metamath C

- MMC is not a “general-purpose” programming language
  - Someday, it can hope to be about as general purpose as C or Rust, but this is a gargantuan effort for many reasons
- The niche MMC fills is writing executable programs which *provably* satisfy some condition
- Most programs don’t need this property, but correctness is important to some degree in almost every program, and (approximate) type correctness is mainstream

Type system → static analysis → theorem prover

Programming languages have come a long way in terms of provable correctness

- Assembly: bare minimum type system required to make instructions compilable

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- **Metamath C**

A type checker is just a simple theorem prover; the study of one naturally leads to the other

Examples: Procedures

- This is a function that takes two 32 bit integers and returns their sum, wrapped to 32 bits

```
proc add2(x: u32, y: u32): u32 {
  return (x + y) as u32;
}
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proc add2(x: u32, y: u32): u32 {
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}
```

- Supports multiple returns and dependent types for writing preconditions and postconditions

```
proc deptypes(x: u32, _: x = 0): y: u32, sn((x + y) as u32) {
  1, sn((x + 1) as u32)
}
```

Examples: Tuples and pattern matching

- This function constructs and destructs some tuples. The `sn(1), sn(2)` return type says that this function returns exactly the values 1 and 2

```
proc tuples(): sn(1), sn(2) {
  let x: (nat, nat) := (1, 2);
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```
proc tuples(): sn(1), sn(2) {
  let x: (nat, nat) := (1, 3); // <- changed 2 to 3
  let (one, two) := x;
  sn(one), sn(two) // type error!
}
```

Examples: Control flow

- After an `if` statement, you can capture the property's truth value in a variable:

```
proc if_statement(x: nat) {
  if h: x < 10 {
    // x: nat, h: x < 10
  } else {
    // x: nat, h: ~(x < 10)
  }
}
```

Examples: Control flow

- While loops and assignment:

```
proc while_loop() {
  let b := true;
  let h2 := while h: b {
    // h: b
    b <- false;
  };
  // h2: ~b
}
```

Examples: Numeric types

- There are various fixed size integral types, as well as unbounded integer types

$$\tau ::= \texttt{u8} \mid \texttt{u16} \mid \texttt{u32} \mid \texttt{u64} \mid \texttt{nat} \mid \texttt{i8} \mid \texttt{i16} \mid \texttt{i32} \mid \texttt{i64} \mid \texttt{int} \mid \ldots$$

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  - `cast(x + y, h): u32`: prove that $x + y < 2^{32}$
  - `cast(x + y): u32`: assert that $x + y < 2^{32}$ and crash otherwise

Separation logic

MMC's type system includes the basic primitives of separation logic, for expressing complex properties:

Type	Concrete syntax	Typehood predicate $\boxed{a : -}$	Meaning
$\exists x : \tau_1, \tau_2(x)$	`(ex x:` τ_1`,` $\tau_2(x)$`)`	$\exists x : \tau_1, \boxed{a : \tau_2(x)}$	Existential quantification
$\forall x : \tau_1, \tau_2(x)$	`all x:` τ_1`.` $\tau_2(x)$	$\forall x : \tau_1, \boxed{a : \tau_2(x)}$	Universal quantification
$\tau_1 \to \tau_2$	τ_1 `->` τ_2	$\boxed{a : \tau_1} \to \boxed{a : \tau_2}$	Non-separating implication
$\tau_1 \mathrel{-\!\!*} \tau_2$	τ_1 `-*` τ_2	$\boxed{a : \tau_1} \mathrel{-\!\!*} \boxed{a : \tau_1}$	Separating imp. (magic wand)
$\tau_1 \land \tau_2$	τ_1 `&&` τ_2	$\boxed{a : \tau_1} \land \boxed{a : \tau_2}$	Non-separating conjunction
$\tau_1 * \tau_2$	(τ_1, τ_2)	$\boxed{a.0 : \tau_1} * \boxed{a.1 : \tau_2}$	Separating conjunction
$\tau_1 \lor \tau_2$	τ_1 `||` τ_2	$\boxed{a : \tau_1} \lor \boxed{a : \tau_2}$	Disjunction
$\neg\tau$	`~`τ_1	$\neg\boxed{a : \tau}$	Negation
$\ell \mapsto v$	ℓ `|->` v	$\ell \mapsto v$	Points-to assertion
$\boxed{e : \tau}$	`[`e`:` τ`]`	$\boxed{e : \tau}$	Typing assertion
$\|\tau\|$	`moved(`τ`)`	$\left\|\boxed{a : \tau}\right\|$	Persistent core of τ

The main function

- The theorem to be proved by the MMC compiler depends on the return type of the `main()` function:

```
proc main(): collatz_conjecture {
  // if this program succeeds, then the collatz conjecture is true
  assert(false) // ...not that I know how to write such a program!
}
```

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It is still a research project at this point, but I have every intention to grow this to an industrial strength project eventually.

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- What this language brings to the table is to take those high level proofs and lower them to fully formal proofs about the resulting assembly code

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- The MMC language design is similar to a programming language with contracts like Dafny / Why3, but unlike these the proofs aren't just "skin deep", they are synthesized into a full proof at the low level, through the entire compiler
- It still remains to be seen if these kind of languages are actually usable in practice, but it could be a game-changer, bringing the task of writing formally verified programs down to the level of the average proof assistant user.

Resources

- Metamath: `http://us.metamath.org/`
- Metamath Zero: `https://github.com/digama0/mm0`
- MM0 Youtube tutorial: `https://youtu.be/A7WfrW7-ifw`
- MM0 thesis: `https://digama0.github.io/mm0/thesis.pdf`
- Lean/mathlib: `http://leanprover-community.github.io/`
- Lean Zulip chat: `https://leanprover.zulipchat.com/`
    - Ask me anything on Zulip, I'm there a lot

Thanks!