



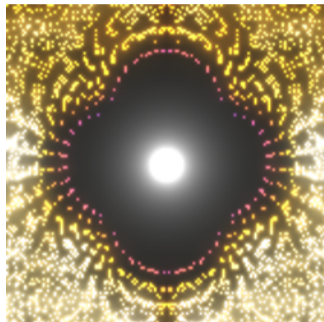
Metamath and Metamath Zero

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February 7, 2023

Who am I?



Github: digama0
Zulip: Mario Carneiro

- ▶ Postdoc in Logic at CMU
- ▶ Proof engineering since 2013
 - ▶ Metamath (maintainer)
 - ▶ Lean 3, Lean 4 mathlib (maintainer)
 - ▶ Dabbled in Isabelle, HOL Light, Coq, Mizar
 - ▶ Metamath Zero (author)
- ▶ Proved 37 of Freek's 100 theorems list in Metamath
- ▶ Lots of library code in `set.mm` and `mathlib`
- ▶ My PhD thesis was about Metamath Zero
- ▶ Say hi at <https://leanprover.zulipchat.com>

Part I: Metamath

Metamath is:

- ▶ A computer language for writing mathematical proofs
- ▶ A program `metamath.exe` to verify proofs in the Metamath language
- ▶ A library of completed proofs in a wide variety of axiomatic systems
 - ▶ `set.mm`: Over 40000 proofs deriving consequences of ZFC
 - ▶ Covers material in set theory, category theory, real analysis, calculus, number theory, algebra, topology, linear algebra, lattice theory, graph theory
 - ▶ 74 from Freek Wiedijk's 100 theorems list, which puts it 4th on the list behind HOL Light, Isabelle, and Coq
 - ▶ `iset.mm`: 10000 proofs in intuitionistic ZF
 - ▶ `nf.mm`: 5900 proofs in NF set theory
 - ▶ `ql.mm`: 1100 proofs in quantum logic
 - ▶ Other databases: `hol.mm`, `dtl.mm`, `peano.mm`, `miu.mm`

Metamath looks like: (set.mm)

```
00344
66523 $( Function with a domain of two different values. (Contributed by FL,
66524 26-Jun-2011.) (Revised by Mario Carneiro, 26-Apr-2015.) $)
66525 fnprg $p ⊢ ( ( ( A e. V ∧ B e. W ) ∧ ( C e. X ∧ D e. Y ) ∧ A ≠ B )
66526 → { <. A , C > . , <. B , D > . } Fn { A , B } ) $=
66527 ( wcel wa wne w3a cop cpr wfun cdm wceq wfn funprg dmprog 3ad2ant2 df-fn
66528 sylanbr ) AEIBFIJZCGIDHIIJZABKZLACMBDMNZOUGPABNZQUGUHRABCDEFHGSUEUUIUFACB
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66537 ( wcel w3a wne cop cdm wceq csu cun ctp wfun wfn funptg wa dmsnopg 3ad2ant1
66538 cpr 3ad2ant2 jca uneq12 syl df-pr syl6eqr dmeqi eqeq1i dmun sylbr 3ad2ant3
66539 bitri uneq12d df-tp eqtri 3eqtr4g df-fn sylanbr ) JDMKHMILMNZAEMZBFMZCGMZN
66540 ZJKOJLOKLONZNZJAPZKBPZLCPZUAZUBVQZJKLUAZRVQVSUCABCDEFHGIJKLUDVMVNVUOHZQZVP
66541 SZQZTZJKUHZLSZTVRVSMVAWEWCWFMVNSZQZVOSZQZTZWERZWAWERZVMWKJSZKSZTZWEVMMHWN
66542 RZJWJWORZUEZWKWPRVKVGSVLVKQWQRVHVIVQVJJAUFUGVIVHNRVJKB FUFUIUJUIWHWNJWOUKU
66543 LJKUMUNWMMGWITZQZWERLWAXAWEVTWTVNVUOMUOUPXAKWENGWUQUPUTURVKVGCWFRZVLVJV
66544 HXBVILCGUFUSUIVAVRVTWBTZQWDVQXCVNVQVPUVBUOVTWBUQCJCLVBVDVQVSEVF $.
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66557 ( wne w3a cop ctp wfun cdm wceq wfn funtp dmtpop a1i df-fn sylanbr ) ABM
66558 ACMBCMNZADOBEOCFOPZQUGRABCPZSZUGUHTABCDEFHGIJKLUAIUFADBEFCJKLUBUCUGUHUU
66559 E $.
66560 $}
66561
```

Metamath looks like: (mmj2)

```
ProofAsstGUI - Page502.mmp
File Edit Cancel Unify Search IL GMFF Help
$( <MM> <PROOF_ASST> THEOREM=95ple96 LOC_AFTER=
*
Page502.mmp

If mmj2 is given no reference and no hypotheses, and isn't allowed
to use its automation capabilities, then unsurprisingly mmj2 can't
prove the simple claim that  $95 + 1 = 96$ .
But by adding the '!' prefix, mmj2 was allowed to use its automation
capabilities, and mmj2 quickly created the following proof:

d1::9nn0      |- 9 e. NN0
d2::5nn0      |- 5 e. NN0
d3::5ple6     |- ( 5 + 1 ) = 6
d4::eqid      |- ; 9 5 = ; 9 5
qed:d1,d2,d3,d4:decsuc |- ( ; 9 5 + 1 ) = ; 9 6

* More generally, mmj2 will generally be able to finish a step if you provide:

- Only the ref (mmj2 will create the statement and derive the steps)
- The ref and hvps (mmj2 will create the statement)

I-PA-0119 Theorem 95ple96: RPN-format Metamath proof generated!
-----
```

Metamath looks like: (MPE)



Mirrors > Home > MPE Home > Th_List > ruc

Metamath Proof Explorer

< Previous Next >
Nearby theorems

Structured version Visualization version GIF version

Theorem ruc 14994

Description: The set of positive integers is strictly dominated by the set of real numbers, i.e. the real numbers are uncountable. The proof consists of lemmas [ruclem1](#) 14982 through [ruclem13](#) 14993 and this final piece. Our proof is based on the proof of Theorem 5.18 of [Truss] p. 114. See [ruclem13](#) 14993 for the function existence version of this theorem. For an informal discussion of this proof, see [mmcomplex.html#uncountable](#). For an alternate proof see [rucALT](#) 14961. This is Metamath 100 proof #22. (Contributed by NM, 13-Oct-2004.)

Assertion

Ref	Expression
ruc	$\vdash N < \mathbb{R}$

Proof of Theorem ruc

Step	Hyp	Ref	Expression
1		reex <small>10049</small>	$\vdash \mathbb{R} \in V$
2		nssre <small>11046</small>	$\vdash \mathbb{N} \subseteq \mathbb{R}$
3		ssdomg <small>8023</small>	$\vdash (\mathbb{R} \in V \rightarrow (\mathbb{N} \subseteq \mathbb{R} \rightarrow \mathbb{N} \preccurlyeq \mathbb{R}))$
4	1, 2, 3	mp2 <small>9</small>	$\vdash \mathbb{N} \preccurlyeq \mathbb{R}$
5		ruclem13 <small>14993</small>	$\dots \vdash \neg f: \mathbb{N} \text{-onto} \rightarrow \mathbb{R}$
6		flofo <small>6158</small>	$\dots \vdash (f: \mathbb{N} \text{-1-1-onto} \rightarrow \mathbb{R} \rightarrow f: \mathbb{N} \text{-onto} \rightarrow \mathbb{R})$
7	5, 6	mto <small>188</small>	$\vdash \neg f: \mathbb{N} \text{-1-1-onto} \rightarrow \mathbb{R}$
8	7	nex <small>1731</small>	$\vdash \vdash \neg \exists f f: \mathbb{N} \text{-1-1-onto} \rightarrow \mathbb{R}$
9		bren <small>7866</small>	$\vdash (\mathbb{N} \approx \mathbb{R} \leftrightarrow \exists f f: \mathbb{N} \text{-1-1-onto} \rightarrow \mathbb{R})$
10	8, 9	mtbir <small>313</small>	$\vdash \neg \mathbb{N} \approx \mathbb{R}$
11		brsdom <small>8000</small>	$\vdash (\mathbb{N} < \mathbb{R} \leftrightarrow (\mathbb{N} \preccurlyeq \mathbb{R} \wedge \neg \mathbb{N} \approx \mathbb{R}))$
12	4, 10, 11	mpbir2an <small>955</small>	$\vdash \mathbb{N} < \mathbb{R}$

Colors of variables: wff setvar class

Syntax hints: \neg wfi 3 \exists wex 1704 \in wcel 1890 \forall cvv 3201 \subseteq wss 2576 class class class wbr 4057 -onto wf0 5900 -1-1-onto wflo 5901 \approx cen 7974 \preccurlyeq cdom 7975 $<$ csdm 7976 \mathbb{R} cci 8957 \mathbb{N} cni 11042

This theorem was proved from axioms: [ax-mp](#) 5 [ax-1](#) 6 [ax-2](#) 7 [ax-3](#) 8 [ax-gen](#) 1722 [ax-4](#) 1737 [ax-5](#) 1839 [ax-6](#) 1888 [ax-7](#) 1935 [ax-8](#) 1992 [ax-9](#) 1999 [ax-10](#) 2019 [ax-11](#) 2034 [ax-12](#) 2047 [ax-13](#) 2246 [ax-ext](#) 2603 [ax-sep](#) 4776 [ax-nul](#) 4794 [ax-pow](#) 4848 [ax-pr](#) 4911 [ax-un](#) 6968 [ax-cnex](#) 10014 [ax-resscn](#) 10015 [ax-1cn](#) 10016 [ax-icn](#) 10017 [ax-addcl](#) 10018 [ax-addrcl](#) 10019 [ax-mulcl](#) 10020 [ax-mulrcl](#) 10021 [ax-mulcom](#) 10022 [ax-associ](#) 10023 [ax-mulass](#) 10024 [ax-distr](#) 10025 [ax-i2m1](#) 10026 [ax-1ne0](#) 10027 [ax-1rid](#) 10028 [ax-mnegex](#) 10029 [ax-rrcex](#) 10030 [ax-cnre](#) 10031 [ax-pre-lttri](#) 10032 [ax-pre-ltrn](#) 10033 [ax-pre-ltadd](#) 10034 [ax-pre-mulgt0](#) 10035 [ax-pre-sup](#) 10036

Metamath's good ideas

Metamath is not the most popular theorem prover,
but it has some good ideas that are not shared with its contemporaries.

What makes Metamath unique?

Metamath's good ideas

- ▶ Separate proof authoring from proof checking
- ▶ Have a simple spec for the logical core

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(and ideally fast and not resource-intensive)

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- ▶ Metamath stores *proofs*, not *proof scripts*

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66540 ZJKOJLKLONZJAPZKBPZLCPZUAZUBVQZJKLUAZRVSUCABCDFGHIJKLUDVMVVOUHZQZVP
66541 SZQTZJKUHZLSZTVRVSVMNAWEWCWFMVNSZQZVOSZQTZWERZWAWERZVMNKJSZKSZTZWELVHMHN
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66543 LJKUMUNMMGWTZQZWERLWAXAWEVTVTVVVOUMJOUPIXAKWEWGWVUQUPUTURVKVGMCFRZVLVJV
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66557 ( wne w3a cop ctp wfun cdm wceq wfn funtp dmtop all df-fn sylanbr ) ABM
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66559 E $.
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- ▶ Checking metamath proofs is massively faster than checking Lean, Coq, Isabelle, HOL Light proofs
 - ▶ The classic verifier metamath.exe checks set.mm, a library on the same order of magnitude as Lean mathlib, in 8 seconds

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 - ▶ An optimized metamath verifier has achieved the same feat in 0.9 seconds

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ZJKOJLKLONZJAPZKBPZLCPZUAZUBVQZJKLUAZRVSUCABCDFGHIJKLUDVMVVOUHZQZVP
SZQZTJJKUHZLSZTVRVSVMAWEWCWFVMSVNSZQZVOSZQZTZWZRZWAWEZVMNKJZSKSZTZWZVMHNN
RZWMWORZUEZKNRPRVKVGSVLVKQWRVHVWQVJJAUFUGVIVHWRVJKBFUFUIUJUIHMMWJWOUKU
LJKUMUNMMGWITZQZWERLWAXAWEVTVTVNVOUMJOUPIXAWKWEWGVIUQPUTURVKVGCWFRZVLVJV
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```


Metamath's good ideas

- ▶ Separate proof authoring from proof checking
- ▶ Have a simple spec for the logical core

Have a simple spec for the logical core

- ▶ Metamath has a prose specification in the Metamath book

The next section contains the complete specification of the Metamath language. It serves as an authoritative reference and presents the syntax in enough detail to write a parser and proof verifier. The specification is terse and it is probably hard to learn the language directly from it, but we include it here for those impatient people who prefer to see everything up front before looking at verbose expository material. Later sections explain this material and provide examples. We will repeat the definitions in those sections, and you may skip the next section at first reading and proceed to Section 4.2 (p. 116).

4.1 Specification of the Metamath Language

Sometimes one has to say difficult things, but one ought to say them as simply as one knows how.

G. H. HARDY²

4.1.1 Preliminaries

A Metamath **database** is built up from a top-level source file together with any source files that are brought in through file inclusion commands (see below). The only characters that are allowed to appear in a Metamath source file are the 94 non-whitespace printable ASCII characters, which are digits, upper and lower case letters, and the following 32 special characters:

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; < = > ? @ [ \ ] ^ _ ` { | } ~
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plus the following characters which are the “white space” characters: space (a printable character), tab, carriage return, line feed, and form feed. We will use **typewriter** font to display the printable characters.

A Metamath database consists of a sequence of three kinds of **tokens** separated by **white space** (which is any sequence of one or more white space characters). The set of **keyword** tokens is $\{ \$, \$\}, \$c, \$v, \$f, \$e, \$d, \$a, \$p, \$., \$=, \$(\, \$), \$[, \text{ and } \$\}$. The last four are called **auxiliary** or preprocessing keywords. A **label** token consists of any combination of letters, digits, and the characters hyphen, underscore, and period. A **math symbol** token may consist of any combination of the 93 printable standard ASCII characters other than space or $\$$. All tokens are case-sensitive.

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- ▶ The HTML documentation is full of pages of introductory material which assumes no mathematical background

Theorem `opreq2i` 2368
Description: Equality inference for operations.

Hypothesis

Ref	Expression
<code>opreq1i</code> 1	$A = B$

Assertion

Ref	Expression
<code>opreq2i</code>	$(CFA) = (CFB)$

$A = B$
 $2 = (1 + 1)$ ②

$(CFA) = (CFB)$
 $(2 + 2) = (2 + (1 + 1))$ ③

Proof of Theorem 2p2e4

Step	Hyp	Ref	Expression
1		<code>df-2</code> 2348	$2 = (1 + 1)$
2	1	<code>opreq2i</code> 2368	$(2 + 2) = (2 + (1 + 1))$
3		<code>df-4</code> 2350	$4 = (3 + 1)$

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 - ▶ The main ones in use are MM-PA and mmj2, and another one (metamath-knife) is in development
- ▶ Metamath has also been used for machine learning (Holophrasm, GPT-f)

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Metamath for AI/ML/ATP applications

- ▶ Metamath is a very friendly language for bulk processing, because it has such a simple grammar and few core concepts
 - ▶ In many cases you can get relevant and accurate information about theorem structure using regexes
 - ▶ There is only one kind of proof step (a theorem application), so proofs are just trees of applications and verification is uniform
- ▶ It also has a large body of human-curated mathematics, which is good for training and testing automated provers
- ▶ Verification and processing is quite fast, so the bottleneck is usually the external processing (the ATP, ML training etc)

Part II: Metamath Zero

Deficiencies of Metamath

- ▶ Metamath tries to simultaneously serve the human reader and the computer verifier, but they have divergent needs

```
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60523 $( Function with a domain of two different values. (Contributed by FL,
60524 26-Jun-2011.) (Revised by Mario Carneiro, 26-Apr-2015.) $)
60525 fnprg $p ⊢ ( ( A e. V ∧ B e. W ) ∧ ( C e. X ∧ D e. Y ) ∧ A ≠ B )
60526 → { <. A , C > . , <. B , D > . } Fn { A , B } $=
60527 ( wcel wne w3a cop cpr wfun cdm wceq wfn funprg dmpop 3ad2ant2 df-fn
60528 sylanbr ) AEIBFIJZCGIDHIJZABKZLACBMDNZOUGPABNZQZUGUHRABCDEFHGSUEUDUIUFACB
60529 DGHUAUGUHUBUC $.
60530
60531 $( Function with a domain of three different values. (Contributed by
60532 Alexander van der Vekens, 5-Dec-2017.) $)
60533 fntpg $p ⊢ ( ( X e. U ∧ Y e. V ∧ Z e. W )
60534 ∧ ( A e. F ∧ B e. G ∧ C e. H )
60535 ∧ ( X ≠ Y ∧ X ≠ Z ∧ Y ≠ Z ) )
60536 → { <. X , A > . , <. Y , B > . , <. Z , C > . } Fn { X , Y , Z } $=
60537 ( wcel w3a wne cop cdm wceq csn cun ctp wfun wfn funtpg wa dmsnopg 3ad2ant1
60538 cpr 3ad2ant2 jca uneq12 syl df-pr syl0eqr dmeqi eqeq1i dmun sylibr 3ad2ant3
60539 bitri uneq12d df-tp eqtri 3eqtr4g df-fn sylanbr ) JDMKMLIMNZAEMZBFMZCGMZN
60540 ZJKOJLKLONZJZAPZKBPZLCPZUAZUBVQQZJKLUAZRQVSUCABCOEFGHIJKLUDVMVWVOUHZZQZVP
60541 SQZTZJJKUHZLSZTVRVSVMAWEWCWFVMVNSZQZVOSZQZTZWERZAWERZVMKJZSKSZTZWVMMHN
60542 RZJWJWRZUEZWKWPRKVGWVSVLVKWQVRVHVIMQVJJAUFUGVIVHWRVJKBVFUFUIJUJHWNWJWOUKU
60543 LJKUMUNMNGWITZQZWERLWAXAWEVTVTVNVVOUMJOUXPAWKWEWGWIUQPUTURVYKVGWCWFRZVLJV
60544 HXBVILCGUFUSIVAVRVTWBTZQMDVQXCVNVVOPVBUOVTWBUQCJKLVBVDVQVSEVF $.
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60555 fntp $p ⊢ ( ( A ≠ B ∧ A ≠ C ∧ B ≠ C )
60556 → { <. A , D > . , <. B , E > . , <. C , F > . } Fn { A , B , C } $=
60557 ( wne w3a cop ctp wfun cdm wceq wfn funtp dmpop ali df-fn sylanbr ) ABM
60558 ACBMCNZADOBEOCFOPZQUGRABCPZSZUGUHTABCDEFHGIJKLUAIUFIADFBEFCJKLUBUCUGUHUDDU
60559 E $.
60560
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- ▶ Metamath automation is decentralized
 - ▶ This is nice in principle, but in practice most people won't be writing their own proof assistant

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```
00364
66523 $( Function with a domain of two different values. (Contributed by FL,
66524 26-Jun-2011.) (Revised by Mario Carneiro, 26-Apr-2015.) $)
66525 fnprg $p ⊢ ( ( ( A e. V ∧ B e. W ) ∧ ( C e. X ∧ D e. Y ) ∧ A ≠ B )
66526 → { <. A , C > . , <. B , D > . } Fn { A , B } ) $=
66527 ( wcel wa wne w3a cop cpr wfun cdm wceq wfn funprg dmpoop 3ad2ant2 df-fn
66528 sylanbr ) AEIBFIJZCGIDHJZABKZLACBMDMNZOUGPABNZQZUGUHRABCDEFHGSUEUDUIUFACB
66529 DGHUAUGUHUBUC $.
66530
66531 $( Function with a domain of three different values. (Contributed by
66532 Alexander van der Vekens, 5-Dec-2017.) $)
66533 funtpg $p ⊢ ( ( ( X e. U ∧ Y e. V ∧ Z e. W )
66534 ∧ ( A e. F ∧ B e. G ∧ C e. H )
66535 ∧ ( X ≠ Y ∧ X ≠ Z ∧ Y ≠ Z ) )
66536 → { <. X , A > . , <. Y , B > . , <. Z , C > . } Fn { X , Y , Z } ) $=
66537 ( ( wcel w3a wne cop cdm wceq csn cun ctp wfun wfn funtpg wa dmsnpg 3ad2ant1
66538 cpr 3ad2ant2 jca uneq12 syl df-pr syl0eqr dmeqi egeq1 dmun sylibr 3ad2ant3
66539 bitri uneq12d df-tp eqtri 3eqtr4g df-fn sylanbr ) JDMKHMIMNZAEMZBFMZCGMZN
66540 ZJKOJLKLONZLNZJAPZKBPZLCPZUAZUBVQQZJKLUAZRQVSUCABCOEFGHIJKLUDVMVWVOUHZZQVP
66541 SZQZTZJKUHZLSZTVRVSVMAWEWCWFVMVNSZQZVOSZQZTZWERZWAERZVMNKJSZKSZTZWEVMHNN
66542 RZMJWORZUEZKWKPRVKGVSVLVKQWRVHVIMQVJJAUFUGVIVHVRVJKBFUFUIUJUIWHMWNJWOUKU
66543 LJKUMUNMMGWITZQZWERLWAXAWEVTVTVVVOUMJOPXAWKWEWGWIUQUPUTURVYKVGWCWFRZVLVJV
66544 HXBV1LCGUFUSUIAVRVTVBWTZQMDVQXCVNVOPVPUVBOVTWBUQCJKLVBVDVQVSVEVF $.
66545
66546
66547 {
66548 fntp.1 $e ⊢ A e. _V $.
66549 fntp.2 $e ⊢ B e. _V $.
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66551 fntp.4 $e ⊢ D e. _V $.
66552 fntp.5 $e ⊢ E e. _V $.
66553 fntp.6 $e ⊢ F e. _V $.
66554 $( A function with a domain of three elements. (Contributed by NM,
66555 14-Sep-2011.) (Revised by Mario Carneiro, 26-Apr-2015.) $)
66556 funtp $p ⊢ ( ( A ≠ B ∧ A ≠ C ∧ B ≠ C )
66557 → { <. A , D > . , <. B , E > . , <. C , F > . } Fn { A , B , C } ) $=
66558 ( wne w3a cop ctp wfun cdm wceq wfn funtp dmpoop ali df-fn sylanbr ) ABM
66559 ACBMCNZAODOEOPFOPZQUGRABCPZSZUGUHTABCDEFHGIJKLUAIUUFADBEFCJKLUBUCUGUHUHU
66560 E $.
66561 }
```

Deficiencies of Metamath

- ▶ Metamath tries to simultaneously serve the human reader and the computer verifier, but they have divergent needs
 - ▶ The big block of compressed proof text is very off-putting for newcomers, and not great for source control either
 - ▶ In practice you need a tool to read proofs
- ▶ Metamath automation is decentralized
 - ▶ This is nice in principle, but in practice most people won't be writing their own proof assistant
 - ▶ Metamath has a reputation for having no automation as a result
 - ▶ Existing MM proof assistants are certainly lacking in small scale automation compared to HOL light, Isabelle, Coq, Lean

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- ▶ Give it a more modern-looking syntax
 - ▶ shamelessly borrowed from Lean

Introduction to Metamath C

Software without bugs is possible

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- Software correctness can be proved by mathematical proof (“deductive verification”)

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Metamath C is a language for writing verified programs.

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 - ▶ The MMC compiler produces MM0 proofs
 - ▶ The MM0 verifier is written in MMC² – bootstrap!

²Currently the MMC verifier for MM0 is not finished, but there are multiple other MM0 verifiers so it can already be used without the bootstrap.

A simple MM0 file: propositional logic

```
delimiter $ ( ~ $ $ ) $;  
strict provable sort wff;  
term im (a b: wff): wff; infixr im: $->$ prec 25;  
term not (a: wff): wff; prefix not: $~$ prec 40;  
  
-- The Lukasiewicz axioms for propositional logic  
axiom ax_1 (a b: wff): $ a -> b -> a $;  
axiom ax_2 (a b c: wff):  
  $ (a -> b -> c) -> (a -> b) -> a -> c $;  
axiom ax_3 (a b: wff):  
  $ (~a -> ~b) -> b -> a $;  
axiom ax_mp (a b: wff):  
  $ a -> b $ >  
  $ a $ >  
  $ b $;  
  
-- Assert that 'P -> P' is provable  
theorem id (P: wff): $ P -> P $;
```

Peano arithmetic

```
... -- predicate logic

--| The sort of natural numbers, or nonnegative integers.
sort nat;

--| '0' is a natural number.
term d0: nat; prefix d0: $0$ prec max;

--| The successor operation: 'suc n' is a natural number when 'n' is.
term suc (n: nat): nat;

--| Zero is not a successor. Axiom 1 of Peano Arithmetic.
axiom sucne0 (a: nat): $ suc a != 0 $;

--| The successor function is injective. Axiom 2 of Peano Arithmetic.
axiom sucinj (a b: nat): $ suc a = suc b <-> a = b $;

--| The induction axiom of Peano Arithmetic. If 'p(0)' is true,
--| and 'p(x)' implies 'p(suc x)' for all 'x', then 'p(x)' is true for all 'x'.
axiom induction {x: nat} (p: wff x):
  $ [ 0 / x ] p -> A. x (p -> [ suc x / x ] p) -> A. x p $;
```


Peano arithmetic

```
--| Addition of natural numbers, a primitive term constructor in PA.  
term add (a b: nat): nat; infixl add: $+$ prec 64;  
--| Multiplication of natural numbers, a primitive term constructor in PA.  
term mul (a b: nat): nat; infixl mul: $*$ prec 70;  
  
--| Addition respects equality.  
axiom addeq (a b c d: nat): $ a = b -> c = d -> a + c = b + d $;  
--| Multiplication respects equality.  
axiom muleq (a b c d: nat): $ a = b -> c = d -> a * c = b * d $;  
--| The base case in the definition of addition.  
axiom add0 (a: nat): $ a + 0 = a $;  
--| The successor case in the definition of addition.  
axiom addS (a b: nat): $ a + suc b = suc (a + b) $;  
--| The base case in the definition of multiplication.  
axiom mul0 (a: nat): $ a * 0 = 0 $;  
--| The successor case in the definition of multiplication.  
axiom mulS (a b: nat): $ a * suc b = a * b + a $;
```

Peano arithmetic

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We define:

- ▶ Propositional logic
- ▶ Predicate logic
- ▶ Class theory
- ▶ $+$, $-$, $*$, $/$, mod, gcd
- ▶ even, odd, disjoint sums
- ▶ ordered pairs, cartesian product
- ▶ finite functions, class functions
- ▶ Integers: $+$, $-$, $*$, $/$, mod
- ▶ Bitwise operators
- ▶ Recursion, exponentiation
- ▶ Lists
- ▶ Set operators
- ▶ finite sets, finite set theory
- ▶ cardinality
- ▶ List ops: length, append, repeat, reverse, map, join, filter, zip, ...

Metaprogramming with MM1

- ▶ MM1 comes with a metaprogramming language based on Scheme

```
do {  
  (display "hello world")      -- hello world  
  {2 + 2}                      -- 4  
  (def x 5)  
  {x + x}                      -- 10  
  (def (f y) {y + y})  
  (f 3)                        -- 6  
  (def (fact x)  
    (if {x = 0}  
        1  
        {x * (fact {x - 1})}))  
  (fact 5)                    -- 120  
};
```

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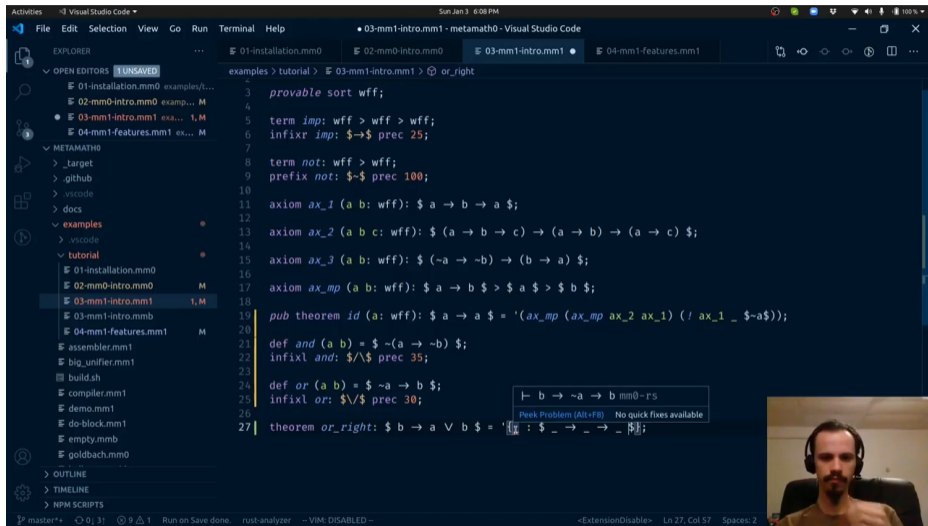
```
theorem _: $ ,19 * ,120 + ,2 = ,2282 $ = norm_num;
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- ▶ `theorem _` means it is an example
- ▶ `norm_num` is the tactic which proves the theorem
- ▶ `,19` calls a preprocessor to render 19 as a term. The actual theorem proved is:

```
theorem _: $ (x1 :x x3) * (x7 :x x8) + x2 = (x8 :x xe :x xa) $;
```

that is, $0x13 \cdot 0x78 + 0x2 = 0x8ea$ which is the theorem written in hexadecimal


Working with MM1



The screenshot shows the Visual Studio Code interface with a file explorer on the left and a code editor on the right. The file explorer shows a project structure with folders like 'examples', 'METAMATH0', and 'tutorial'. The code editor displays a Metamath file named '03-mm1-intro.mm1' with the following content:

```
examples > tutorial > 03-mm1-intro.mm1 > or_right
1
2
3   provable sort wff;
4
5   term imp: wff > wff > wff;
6   infixr imp: $->$ prec 25;
7
8   term not: wff > wff;
9   prefix not: $~$ prec 100;
10
11  axiom ax_1 (a b: wff): $ a → b → a $;
12
13  axiom ax_2 (a b c: wff): $ (a → b → c) → (a → b) → (a → c) $;
14
15  axiom ax_3 (a b: wff): $ (~a → ~b) → (b → a) $;
16
17  axiom ax_mp (a b: wff): $ a → b $ > $ a $ > $ b $;
18
19  pub theorem id (a: wff): $ a → a $ = '(ax_mp (ax_mp ax_2 ax_1) (! ax_1 _ $~a$))';
20
21  def and (a b) = $ ~(a → ~b) $;
22  infixl and: $/\$ prec 35;
23
24  def or (a b) = $ ~a → b $;
25  infixl or: $V/$ prec 30;
26
27  theorem or_right: $ b → a ∨ b $ = '(! : $ _ → _ → _ $)];
```

A tooltip is visible over the code on line 27, containing the text: $\vdash b \rightarrow \sim a \rightarrow b$ mm0-rs. Below the tooltip, it says "Peek Problem [Alt+FR] No quick fixes available".



From “Metamath Zero (MM0/MM1) tutorial”, <https://youtu.be/A7Wfrw7-ifw>

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 - ▶ This requires a model of the registers, instruction pointer, flags, memory, page permissions, exception state

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- ▶ The ELF file format (the linux equivalent of .exe)

Metamath C

- ▶ This is everything we need to state the correctness theorem for a compiled program:

Program correctness

Program P is correct to specification T if for every initial state $s \in \text{init}(P)$, all nondeterministic evaluations do not cause undefined behavior, and after reaching a final state $s \rightsquigarrow^* s'$, if s' is a successful exit state and $\text{input_consumed}(s') = I$ and $\text{output_produced}(s') = O$, then $T(I, O)$ is true.

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- ▶ Red: definitions from `x86.mm0`
- ▶ Blue: the user specification
- ▶ The Metamath C compiler produces theorems of this form.

Metamath C

- ▶ MMC is not a “general-purpose” programming language
 - ▶ Someday, it can hope to be about as general purpose as C or Rust, but this is a gargantuan effort for many reasons
- ▶ The niche MMC fills is writing executable programs which *provably* satisfy some condition
- ▶ Most programs don't need this property, but correctness is important to some degree in almost every program, and (approximate) type correctness is mainstream

Type system → static analysis → theorem prover

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A type checker is just a simple theorem prover; the study of one naturally leads to the other

Examples: Procedures

- ▶ This is a function that takes two 32 bit integers and returns their sum, wrapped to 32 bits

```
proc add2(x: u32, y: u32): u32 {  
    return (x + y) as u32;  
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- ▶ Supports multiple returns and dependent types for writing preconditions and postconditions

```
proc deptypes(x: u32, _: x = 0): y: u32, sn((x + y) as u32) {  
  1, sn((x + 1) as u32)  
}
```

Examples: Tuples and pattern matching

- ▶ This function constructs and destructs some tuples. The `sn(1)`, `sn(2)` return type says that this function returns exactly the values 1 and 2

```
proc tuples(): sn(1), sn(2) {  
  let x: (nat, nat) := (1, 2);  
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proc tuples(): sn(1), sn(2) {  
  let x: (nat, nat) := (1, 3); // <- changed 2 to 3  
  let (one, two) := x;  
  sn(one), sn(two) // type error!  
}
```

Examples: Control flow

- ▶ After an `if` statement, you can capture the property's truth value in a variable:

```
proc if_statement(x: nat) {  
  if h: x < 10 {  
    // x: nat, h: x < 10  
  } else {  
    // x: nat, h: ~(x < 10)  
  }  
}
```

Examples: Control flow

- ▶ While loops and assignment:

```
proc while_loop() {  
  let b := true;  
  let h2 := while h: b {  
    // h: b  
    b <- false;  
  };  
  // h2: ~b  
}
```

Examples: Numeric types

- ▶ There are various fixed size integral types, as well as unbounded integer types

$\tau ::= \text{u8} \mid \text{u16} \mid \text{u32} \mid \text{u64} \mid \text{nat} \mid \text{i8} \mid \text{i16} \mid \text{i32} \mid \text{i64} \mid \text{int} \mid \dots$

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 - ▶ $(x + y)$ **as u32**: wrap the result
 - ▶ $(x + y)$: **u32**: make the type checker prove it is in range (usually only works if the values of x and y are known)
 - ▶ **cast** $(x + y, h)$: **u32**: prove that $x + y < 2^{32}$
 - ▶ **cast** $(x + y)$: **u32**: assert that $x + y < 2^{32}$ and crash otherwise

Separation logic

MMC's type system includes the basic primitives of separation logic, for expressing complex properties:

Type	Concrete syntax	Typehood predicate $a : -$	Meaning
$\exists x : \tau_1, \tau_2(x)$	ex $x : \tau_1, \tau_2(x)$	$\exists x : \tau_1, a : \tau_2(x)$	Existential quantification
$\forall x : \tau_1, \tau_2(x)$	all $x : \tau_1. \tau_2(x)$	$\forall x : \tau_1, a : \tau_2(x)$	Universal quantification
$\tau_1 \rightarrow \tau_2$	$\tau_1 \rightarrow \tau_2$	$a : \tau_1 \rightarrow a : \tau_2$	Non-separating implication
$\tau_1 * \tau_2$	$\tau_1 * \tau_2$	$a : \tau_1 * a : \tau_1$	Separating imp. (magic wand)
$\tau_1 \wedge \tau_2$	$\tau_1 \&\& \tau_2$	$a : \tau_1 \wedge a : \tau_2$	Non-separating conjunction
$\tau_1 * \tau_2$	(τ_1, τ_2)	$a.0 : \tau_1 * a.1 : \tau_2$	Separating conjunction
$\tau_1 \vee \tau_2$	$\tau_1 \tau_2$	$a : \tau_1 \vee a : \tau_2$	Disjunction
$\neg \tau$	$\sim \tau_1$	$\neg a : \tau$	Negation
$\ell \mapsto v$	$\ell \mapsto v$	$\ell \mapsto v$	Points-to assertion
$e : \tau$	$[e : \tau]$	$e : \tau$	Typing assertion
$ \tau $	moved (τ)	$\boxed{a : \tau}$	Persistent core of τ

The main function

- ▶ The theorem to be proved by the MMC compiler depends on the return type of the `main()` function:

```
proc main(): collatz_conjecture {  
    // if this program succeeds, then the collatz conjecture is true  
    assert(false) // ...not that I know how to write such a program!  
}
```

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It is still a research project at this point, but I have every intention to grow this to an industrial strength project eventually.

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- ▶ What this language brings to the table is to take those high level proofs and lower them to fully formal proofs about the resulting assembly code

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- ▶ It still remains to be seen if these kind of languages are actually usable in practice, but it could be a game-changer, bringing the task of writing formally verified programs down to the level of the average proof assistant user.

Resources

- ▶ Metamath: <http://us.metamath.org/>
- ▶ Metamath Zero: <https://github.com/digama0/mm0>
- ▶ MM0 Youtube tutorial: <https://youtu.be/A7WfrW7-ifw>
- ▶ MM0 thesis: <https://digama0.github.io/mm0/thesis.pdf>
- ▶ Lean/mathlib: <http://leanprover-community.github.io/>
- ▶ Lean Zulip chat: <https://leanprover.zulipchat.com/>
 - ▶ Ask me anything on Zulip, I'm there a lot

Thanks!